Some Harder trigonometry problems

1. Let $\tan x \tan 2x + \tan 2x \tan 3x + \cdots. + \tan 6x \tan 7x = 10$, find the value of $\frac{\tan 7x}{\tan x}$.

2. Let $\tan 2x = \frac{2t\sqrt{1-t^2}}{1-2t^2}$, and assume $0 < t < \frac{1}{\sqrt{2}}$. Find $\sin x$.

3. Compute $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}$.

4. **(a)** By using induction on $2(x^n + y^n) \geq (x^{n-1} + y^{n-1})(x + y)$, where x, y ≥ 0 .

prove the Power Mean inequality: $\frac{x^n+y^n}{2} \geq \left(\frac{x+y}{2}\right)^n$.

(b) Prove that $\frac{1}{2^{n-1}} \leq \sin^{2n} x + \cos^{2n} x \leq 1, \ \forall x \in \mathbf{R}$.

5. If $\frac{x}{\tan(\theta+\alpha)} = \frac{y}{\tan(\theta+\beta)} = \frac{z}{\tan(\theta+\gamma)}$, find the value of :

$$E = \left(\frac{x+y}{x-y}\right)\sin^2(\alpha-\beta) + \left(\frac{y+z}{y-z}\right)\sin^2(\beta-\gamma) + \left(\frac{z+x}{z-x}\right)\sin^2(\gamma-\alpha)$$

6. Let $\begin{cases} \cos x - \cos y = a \\ \cos 3x - \cos 3y = b \end{cases}$

Find the value of

$E = \cos 2x + \cos 2y + 2\cos x \cos y$

7. Let $T_n = \sin^n x + \cos^n x$

Prove that $6T_{10} - 15T_8 + 10T_6 = 1$.

8. Let $(1-k)\tan^2\left(\frac{a}{2}\right) = (1+k)\tan^2\left(\frac{b}{2}\right)$, find the value of

$E = k^2 + (1 + k\cos a)(1 - k\cos b)$.

9. Compute : $\sec^2\frac{\pi}{7} + \sec^2\frac{2\pi}{7} + \sec^2\frac{3\pi}{7}$.

10. Find the sum of the series:

$S = 1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots + 2\cos nx$

1. From given:

$(1-\tan x\tan 2x)+(1-\tan 2x\tan 3x)+\cdots.+(1-\tan 6x\tan 7x)=6-10$

$\frac{\tan 2x-\tan x}{\tan(2x-x)}+\frac{\tan 3x-\tan 2x}{\tan(3x-2x)}+\cdots+\frac{\tan 7x-\tan 6x}{\tan(7x-6x)}=-4$

$\frac{\tan 2x-\tan x}{\tan\ x}+\frac{\tan 3x-\tan 2x}{\tan\ x}+\cdots+\frac{\tan 7x-\tan 6x}{\tan\ x}=-4$

$\frac{\tan 7x-\tan x}{\tan\ x}=-4$

$\tan 7x-\tan x=-4\tan x$

$\tan 7x=-3\tan x$

$\therefore\ \frac{\tan 7x}{\tan x}=-3$

2. **Method 1**

$\tan 2x=\frac{2t\sqrt{1-t^2}}{1-2t^2}\Longrightarrow \tan^2 2x=\frac{4t^2(1-t^2)}{(1-2t^2)^2}$

$\Longrightarrow \sec^2 2x=1+\tan^2 2x=1+\frac{4t^2(1-t^2)}{(1-2t^2)^2}=\frac{(1-2t^2)^2+4t^2(1-t^2)}{(1-2t^2)^2}=\frac{(1-4t^2+16t^4)+(4t^2-16t^4)}{(1-2t^2)^2}=\frac{1}{(1-2t^2)^2}$

$\Longrightarrow \sec 2x=\frac{1}{1-2t^2}$, since $0<t<\frac{1}{\sqrt{2}}$, we may neglect the negative root .

$\Longrightarrow \cos 2x=1-2t^2$

Now, $\sin^2 x=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}[1-(1-2t^2)]=t^2$

$\therefore\ \sin x=t$

Method 2

Let $\sin x=u$, than $\tan x=\frac{u}{\sqrt{1-u^2}}$

$\tan 2x=\frac{2t\sqrt{1-t^2}}{1-2t^2}\Longrightarrow\frac{2\tan x}{1-\tan^2 x}=\frac{2t\sqrt{1-t^2}}{1-2t^2}\Longrightarrow\frac{2\frac{u}{\sqrt{1-u^2}}}{1-\left(\frac{u}{\sqrt{1-u^2}}\right)^2}=\frac{2t\sqrt{1-t^2}}{1-2t^2}\Longrightarrow\frac{2u\sqrt{1-u^2}}{1-2u^2}=\frac{2t\sqrt{1-t^2}}{1-2t^2}$

$\therefore\quad u=\sin x=t$

3. Obvious that $\sin 0,\sin\frac{\pi}{7},\sin\frac{2\pi}{7},\sin\frac{3\pi}{7},\sin\frac{4\pi}{7},\sin\frac{5\pi}{7},\sin\frac{6\pi}{7}$ are roots of the equation

$\sin 7x=0.$

$\cos 7x+i\sin 7x=(\cos x+i\sin x)^7=(c+is)^7$

$=(c^7-21c^5s^2+35c^3s^4-7cs^7)+i(7c^6s-35c^4s^3+21c^2s^5-s^7)$, by Binomial Theorem

Compare imaginary parts,

$\sin 7x = 7c^6 s - 35c^4 s^3 + 21c^2 s^5 - s^7 = 7(1-s^2)^3 s - 35(1-s^2)^2 s^3 + 21(1-s^2)s^5 - s^7$

$= 7s - 56s^3 + 112s^5 - 64s^7$, where $s = \sin x$

$\sin 7x = 0 \Longrightarrow 7s - 56s^3 + 112s^5 - 64s^7 = 0 \Longrightarrow s(64s^6 - 112s^4 + 56s^2 - 7) = 0$

$\sin 0, \sin\frac{\pi}{7}, \sin\frac{2\pi}{7}, \sin\frac{3\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{5\pi}{7}, \sin\frac{6\pi}{7}$ are roots of $s(64s^6 - 112s^4 + 56s^2 - 7) = 0$

$\sin\frac{\pi}{7}, \sin\frac{2\pi}{7}, \sin\frac{3\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{5\pi}{7} \sin\frac{6\pi}{7}$ are roots of $64s^6 - 112s^4 + 56s^2 - 7 = 0$

Product of roots $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7} = -\frac{-7}{64} = \frac{7}{64}$

Since $\sin\frac{\pi}{7} = \sin\frac{6\pi}{7}, \sin\frac{2\pi}{7} = \sin\frac{5\pi}{7}, \sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}$

We have $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sqrt{\frac{7}{64}} \approx 0.3307189138831$

4. **(a)** Let P(n): $2(x^n + y^n) \geq (x^{n-1} + y^{n-1})(x+y)$, where x, y ≥ 0 .

For P(1), $2(x+y) \geq (x^0 + y^0)(x+y)$

For P(2), $2(x^2+y^2) \geq (x+y)^2 + (x-y)^2 \geq (x+y)^2 = (x^{2-1} + y^{2-1})(x+y)$

Assume P(k - 1) and P(k) are true for some k∈**N** .

That is, $2\left(x^{k-1} + y^{k-1}\right) \geq \left(x^{k-2} + y^{k-2}\right)(x+y)$

$2\left(x^k + y^k\right) \geq \left(x^{k-1} + y^{k-1}\right)(x+y)$

For P(k + 1),

$2\left(x^{k+1} + y^{k+1}\right) = 2\left(x^k + y^k\right)(x+y) - 2xy\left(x^{k-1} + y^{k-1}\right)$

$\geq \left(x^{k-1} + y^{k-1}\right)(x+y)(x+y) - xy\left(x^{k-2} + y^{k-2}\right)(x+y)$

$= \left[\left(x^{k-1} + y^{k-1}\right)(x+y) - xy\left(x^{k-2} + y^{k-2}\right)\right](x+y)$

$= \left[\left(x^k + y^k\right) + x^{k-1}y + xy^{k-1} - x^{k-1}y - xy^{k-1}\right](x+y)$

$= \left(x^k + y^k\right)(x+y)$ and P(k + 1) is true.

By the second principle of Mathematical Induction, P(n) is true ∀n∈**N**.

For Power Mean inequality, from above

$\frac{x^n+y^n}{2} \geq \frac{1}{2}(x^{n-1} + y^{n-1})\left(\frac{x+y}{2}\right) \geq \frac{1}{2}(x^{n-2} + y^{n-2})\left(\frac{x+y}{2}\right)^2$

$\geq \cdots \geq \frac{1}{2}(x^0 + y^0)\left(\frac{x+y}{2}\right)^n = \left(\frac{x+y}{2}\right)^n$, by deduction, hence induction.

(b) (i) Since $0 \leq \sin^2 x, \cos^2 x \leq 1$

$\sin^{2n} x \leq \sin^2 x$

$\cos^{2n} x \leq \cos^2 x$

Adding, we get $\sin^{2n} x + \cos^{2n} x \leq \sin^2 x + \cos^2 x = 1$

(ii) By the Power Mean inequality in (a),

$$\sin^{2n} x + \cos^{2n} x \geq 2\left(\frac{\sin^2 x + \cos^2 x}{2}\right)^n = \frac{1}{2^{n-1}}$$

5. Let $\frac{x}{\tan(\theta+\alpha)} = \frac{y}{\tan(\theta+\beta)} = \frac{z}{\tan(\theta+\gamma)} = k$

Then $\frac{x+y}{x-y} = \frac{k\tan(\theta+\alpha)+k\tan(\theta+\beta)}{k\tan(\theta+\alpha)-k\tan(\theta+\beta)} = \frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)} = \frac{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}+\frac{\sin(\theta+\beta)}{\cos(\theta+\beta)}}{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}-\frac{\sin(\theta+\beta)}{\cos(\theta+\beta)}}$

$= \frac{\sin(\theta+\alpha)\cos(\theta+\beta)+\cos(\theta+\alpha)\sin(\theta+\beta)}{\sin(\theta+\alpha)\cos(\theta+\beta)-\cos(\theta+\alpha)\sin(\theta+\beta)} = \frac{\sin[(\theta+\alpha)+(\theta+\beta)]}{\sin[(\theta+\alpha)-(\theta+\beta)]} = \frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$

$\therefore \left(\frac{x+y}{x-y}\right)\sin^2(\alpha-\beta) = \sin(2\theta+\alpha+\beta)\ \sin(\alpha-\beta) = -\frac{1}{2}[\cos(\theta+\alpha) - \cos(\theta+\beta)]$

Similarly, $\left(\frac{y+z}{y-z}\right)\sin^2(\beta-\gamma) = -\frac{1}{2}[\cos(\theta+\beta) - \cos(\theta+\gamma)]$

$\left(\frac{z+x}{z-x}\right)\sin^2(\gamma-\alpha) = -\frac{1}{2}[\cos(\theta+\gamma) - \cos(\theta+\alpha)]$

$\therefore\ E = \left(\frac{x+y}{x-y}\right)\sin^2(\alpha-\beta) + \left(\frac{y+z}{y-z}\right)\sin^2(\beta-\gamma) + \left(\frac{z+x}{z-x}\right)\sin^2(\gamma-\alpha) = 0$

6. $\cos 3x - \cos 3y = b$

$(4\cos^3 x - 3\cos x) - (4\cos^3 y - 3\cos y) = b$

$4(\cos^3 x - \cos^3 y) - 3(\cos x - \cos y) = b$

$4(\cos x - \cos y)(\cos^2 x + \cos x\cos y + \cos^2 y) - 3a = b$

$4a\left(\frac{1+\cos 2x}{2} + \cos x\cos y + \frac{1+\cos 2y}{2}\right) - 3a = b$

$2a(1 + \cos 2x + \cos 2y + 2\cos x\cos y) - 3a = b$

$2a(1 + E) - 3a = b$

$\therefore\ E = \frac{a+b}{2a}$

7. Let $s = \sin x$, $c = \cos x$, $T_n = \sin^n x + \cos^n x = s^n + c^n$

$T_2 = s^2 + c^2 = 1$

$T_4 = (s^2 + c^2)^2 - 2s^2c^2 = 1 - 2s^2c^2$

$T_6 = (s^4 + c^4)(s^2 + c^2) - s^2c^2(s^2 + c^2) = (1 - 2s^2c^2) - s^2c^2 = 1 - 3s^2c^2$

$T_8 = (s^6 + c^6)(s^2 + c^2) - s^2c^2(s^4 + c^4) = (1 - 3s^2c^2) - s^2c^2(1 - 2s^2c^2) = 1 - 4s^2c^2 + 2s^4c^4$

$T_{10} = (s^8 + c^8)(s^2 + c^2) - s^2c^2(s^6 + c^6) = 1 - 4s^2c^2 + 2s^4c^4 - s^2c^2(1 - 3s^2c^2)$

$= 1 - 5s^2c^2 + 5s^4c^4$

$\therefore\ 6T_{10} - 15T_8 + 10T_6$

$= 6(1 - 5s^2c^2 + 5s^4c^4) - 15(1 - 4s^2c^2 + 2s^4c^4) + 10(1 - 3s^2c^2)$

$= 1$

8. Let $t = \tan\left(\frac{a}{2}\right)$, $u = \tan\left(\frac{b}{2}\right)$, then $\cos a = \frac{1-t^2}{1+t^2}$, $\cos b = \frac{1-u^2}{1+u^2}$

Also, $(1-k)\tan^2\left(\frac{a}{2}\right) = (1+k)\tan^2\left(\frac{b}{2}\right) \Rightarrow (1-k)t^2 = (1+k)u^2$ .... (*)

$$E = k^2 + (1 + k\cos a)(1 - k\cos b) = k^2 + \left(1 + k\frac{1-t^2}{1+t^2}\right)\left(1 - k\frac{1-u^2}{1+u^2}\right)$$

$$= k^2 + \frac{[1+t^2+k(1-t^2)][1+u^2-k(1-u^2)]}{(1+t^2)(1+u^2)} = k^2 + \frac{[1+k+(1-k)t^2][1-k+(1+k)u^2]}{(1+t^2)(1+u^2)}$$

$$= k^2 + \frac{[1+k+(1+k)u^2][1-k+(1-k)t^2]}{(1+t^2)(1+u^2)}, \quad \text{by (*)}$$

$$= k^2 + \frac{[(1+k)(1+u^2)][(1-k)(1+t^2)]}{(1+t^2)(1+u^2)} = k^2 + (1+k)(1-k) = 1$$

9. $\cos 7x + i\sin 7x = (\cos x + i\sin x)^7 = (c + is)^7$
$= (c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6) + i(7c^6s - 35c^4s^3 + 21c^2s^5 - s^7)$, by Binomial Theorem

Compare imaginary parts, $\sin 7x = 7c^6s - 35c^4s^3 + 21c^2s^5 - cs^7$
Compare real parts, $\cos 7x = c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6$

$$\tan 7x = \frac{\sin 7x}{\cos 7x} = \frac{7c^6s-35c^4s^3+21c^2s^5-cs^7}{c^7-21c^5s^2+35c^3s^4-7cs^6} = \frac{7t-35t^3+21t^5-t^7}{1-21t^2+35t^4-7t^6}$$, where $t = \tan x$.

$\tan 0, \tan\frac{\pi}{7}, \tan\frac{2\pi}{7}, \tan\frac{3\pi}{7}, \tan\frac{4\pi}{7}, \tan\frac{5\pi}{7}, \tan\frac{6\pi}{7}$ are roots of the equation $\tan 7x = 0$,

Or $7t - 35t^3 + 21t^5 - t^7 = 0$

$\tan\frac{\pi}{7}, \tan\frac{2\pi}{7}, \tan\frac{3\pi}{7}, \tan\frac{4\pi}{7}, \tan\frac{5\pi}{7}, \tan\frac{6\pi}{7}$ are roots of the equation

$t^6 - 21t^4 + 35t^2 - 7 = 0$, where the root $t = 0$ is neglected.

Sum of roots = $\sum \tan\frac{\pi}{7} = 0$

Sum of pair of roots = $\sum \tan\frac{\pi}{7}\tan\frac{2\pi}{7} = -21$

$$\sum \tan^2\frac{\pi}{7} = \left(\sum \tan\frac{\pi}{7}\right)^2 - 2\sum \tan\frac{\pi}{7}\tan\frac{2\pi}{7} = 0^2 - 2(-21) = 42$$

$$\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} + \left(-\tan\frac{3\pi}{7}\right)^2 + \left(-\tan\frac{5\pi}{7}\right)^2 + \left(-\tan\frac{\pi}{7}\right)^2 = 42$$

$$\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} = \frac{42}{2} = 21$$

$$\therefore \quad \sec^2\frac{\pi}{7} + \sec^2\frac{2\pi}{7} + \sec^2\frac{3\pi}{7} = \left(1 + \tan^2\frac{\pi}{7} + 1 + \tan^2\frac{2\pi}{7} + 1 + \tan^2\frac{3\pi}{7}\right) = 3 + 21 = 24$$

10. $S = 1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots + 2\cos nx$

$$S\sin\frac{x}{2} = \sin\frac{x}{2} + 2\cos x\sin\frac{x}{2} + 2\cos 2x\sin\frac{x}{2} + 2\cos 3x\sin\frac{x}{2} + \cdots + 2\cos nx\sin\frac{x}{2}$$

$$S\sin\frac{x}{2} = \sin\frac{x}{2} + \left(\sin\frac{3x}{2} - \sin\frac{x}{2}\right) + \left(\sin\frac{5x}{2} - \sin\frac{3x}{2}\right) + \left(\sin\frac{7x}{2} - \sin\frac{5x}{2}\right) + \cdots + \left(\sin\frac{(2n+1)x}{2} - \sin\frac{(2n-1)x}{2}\right)$$

$$S\sin\frac{x}{2} = \sin\frac{(2n+1)x}{2} - \sin\frac{x}{2} = 2\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}$$

$$\therefore \quad S = \frac{2\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

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